

# INDISPENSABLE HIBI RELATIONS AND GRÖBNER BASES

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**ABSTRACT.** In this paper we consider Hibi rings and Rees rings attached to a poset. We classify the ideal lattices of posets whose Hibi relations are indispensable and the ideal lattices of posets whose Hibi relations form a quadratic Gröbner basis with respect to the rank lexicographic order. Similar classifications are obtained for Rees rings of Hibi ideals.

## INTRODUCTION

The main purpose of this paper is to classify those distributive lattices with the property that the Hibi relations are indispensable and those with the property that Hibi relations form a Gröbner basis with respect to the rank lexicographic order. To be precise let  $L$  be a finite lattice. Attached to this lattice one defines the so-called Hibi ideal as follows: we fix a field  $K$  and consider the polynomial ring  $T = K[\{z_a : a \in L\}]$  over  $K$  whose variables are indexed by the elements of  $L$ . Then

$$I_L = (z_a z_b - z_{a \wedge b} z_{a \vee b} : a, b \in L).$$

is called the *Hibi relation ideal* of  $L$ . Relations of the form  $z_a z_b - z_{a \wedge b} z_{a \vee b}$  are called *Hibi relations*.

The  $K$ -algebra

$$\mathcal{R}_K[L] = T/I_L$$

is called the *Hibi ring* of  $L$  (over  $K$ ).

We order variables in  $T = K[\{z_a : a \in L\}]$  such that  $z_a < z_b$  if  $\text{rank } a < \text{rank } b$  and call any monomial order induced by this ordering the *rank order*.

In [7], Hibi proved the following fundamental fact which says that the  $K$ -algebra  $\mathcal{R}_K[L]$  is a domain (hence a toric ring) if and only if  $L$  is distributive. In fact Hibi showed that for distributive lattice Hibi relations form the reduced Gröbner basis with respect to the reverse lexicographic order. Even though Hibi relations generate  $I_L$ , they may not be indispensable in the sense of Hibi and Ohsugi [8]. In other words, in general there may exist a minimal set of generators of  $I_L$  consisting of relations other than Hibi relations. The simplest example of such a lattice is the Boolean lattice  $B_3$  which consists of all the subsets of a three element set.

In Theorem 1.6 we give the classification of finite distributive lattices with the property that for  $I_L$  the Hibi relations are indispensable. To describe the result, recall that according to Birkhoff's theorem every finite distributive lattice is isomorphic to the ideal lattice of a finite poset. This poset is uniquely determined by  $L$ . In fact, it is the subposet  $P$  of

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$L$  consisting of join-irreducible elements of  $L$ . Among other equivalent conditions for the property that Hibi relations are indispensable, it is shown in Theorem 1.6 that all poset ideals of  $P$  are generated by at most 2 elements. Another equivalent condition says that  $L$  is a conditionally URC lattice. Modifying the definition of uniquely complemented lattices given by Stanley in [9], we call a lattice  $L$  conditionally uniquely relatively complemented (conditionally URC), if each interval  $[a, b]$  in  $L$  has unique complements provided they exist. Recall that  $c, d \in [a, b]$  are called complements of each other with respect to  $[a, b]$  if  $c \vee d = b$  and  $c \wedge d = a$ . In Theorem 1.7, we observe that a conditionally URC lattice is always distributive. We show in Proposition 1.7 that a URC lattice is isomorphic to a sublattice of  $\mathbb{N}^2$  of the form  $[m]_0 \times [n]_0$ , where  $[k]_0 = \{0, 1, \dots, k\}$ .

Motivated by the paper [1] of Aramova, Herzog and Hibi where it is shown in [1, Theorem 2.5] that the Hibi ring of a finite simple planar distributive lattice has a quadratic Gröbner basis if and only if  $L$  is a chain ladder, we classify in Theorem 2.1 all distributive lattices  $L$  having the property that the reduced Gröbner basis of  $I_L$  consists of Hibi relations. One of the equivalent condition states that  $L$  is a chain ladder without critical corner.

Let  $P = \{p_1, \dots, p_n\}$  be a finite poset and  $L$  be its ideal lattice. In the last section of the paper we study the Gröbner basis of the defining ideal  $J_L$  of the Rees ring of the Hibi ideal  $H_L$ . The Hibi ideal  $H_L$  is defined to be the monomial ideal generated by the monomials  $u_a = \prod_{p_i \in a} x_i \prod_{p_i \notin a} y_i$  in the polynomial ring  $K[x_1, \dots, x_n, y_1, \dots, y_n]$ . In [4], the Gröbner basis of  $J_L$  is described with respect to the rank reverse lexicographic order. The main result of Section 4 is Theorem 3.1 where it is shown that a distributive lattice  $L$  is a URC lattice if and only if the reduced Gröbner basis with respect to natural lexicographic order consists of Hibi relations and special linear relations. This result is used in Corollary 3.4 to study for meet-distributive meet-semilattice  $L$ , the reduced Gröbner basis of  $J_L$  with respect to a lexicographic order.

## 1. HIBI RINGS WITH INDISPENSABLE HIBI RELATIONS

In this section we want to classify all distributive lattices  $L$  with the property that the Hibi relations  $z_a z_b - z_{a \wedge b} z_{a \vee b}$  are indispensable, which means that the Hibi relations appear in each minimal binomial set of generators of  $I_L$ . Before discussing this problem we recall some fundamental facts about Hibi rings.

Let  $L$  be a finite distributive lattice. According to Birkhof's theorem, the distributive lattice  $L$  is isomorphic to the ideal lattice of the subposet  $P$  of  $L$  consisting of all join irreducible elements of  $L$ . Thus we may always view  $L$  as the ideal lattice  $\mathcal{I}(P)$  of a poset  $P$ . Say,  $P = \{p_1, \dots, p_r\}$ , and let  $S = K[x_1, \dots, x_r, y_1, \dots, y_r]$  be the polynomial ring in  $2r$  indeterminate. For each  $a \in L$  we define the monomial

$$(1) \quad u_a = \prod_{p_i \in a} x_i \prod_{p_i \notin a} y_i,$$

and consider the  $K$ -algebra homomorphism

$$\varphi: T \rightarrow S, \quad z_a \mapsto u_a.$$

Then one shows that  $\text{Ker}(\varphi) = I_L$ , where  $I_L = (z_a z_b - z_{a \wedge b} z_{a \vee b} : a, b \in L)$ . Hence  $\mathcal{R}_K[L] \cong K[\{u_a : a \in L\}]$ , which implies  $\mathcal{R}_K[L]$  is a domain. In fact Hibi showed that the Hibi relations form a reduced Gröbner basis of  $\text{Ker}(\varphi)$  with respect to reverse rank lexicographic order, see [7] and [6, Theorem 10.1.3].

Note that a lattice is distributive if and only if it does not contain one of the following sublattices shown in Figure 1.

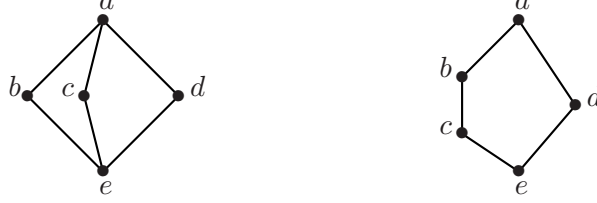


FIGURE 1.

Assume now that  $L$  is not a distributive lattice. Then it contains at least one of the sublattices as shown in Figure 1. Say, it contains the sublattice on the left, then  $z_b z_c - z_a z_e, z_b z_d - z_a z_e \in I_L$ , which implies  $z_b(z_c - z_d) \in I_L$ , but neither  $z_b$  or  $z_c - z_d$  belongs to  $I_L$ . Hence  $I_L$  is not a prime ideal in this case. Similarly it can be seen that  $I_L$  is not prime if  $L$  contains the sublattice on the right.

Distributive lattices are characterized as follows.

**Proposition 1.1.** *Let  $L$  be a lattice. Then the following conditions are equivalent:*

- (a)  *$L$  is a distributive lattice.*
- (b) *Hibi relations form a Gröbner basis with respect to the rank reverse lexicographic order.*

*Proof.* It suffice to proof (b)  $\Rightarrow$  (a): Suppose  $L$  is not a distributive lattice. Then it contains at least one of the sublattices as shown in Figure 1. Say, it contains sublattice on the right, then  $z_b z_d - z_a z_e, z_c z_d - z_a z_e \in I_L$ . Therefore  $f = z_a z_e z_b - z_a z_e z_c \in I_L$ . On the other hand  $\text{in}_<(f) = z_a z_e z_b$  is not divided by any initial term of a Hibi relation in  $I_L$ .  $\square$

Now we come back to the main problem of this section concerning the indispensability of Hibi relations. For example, consider the Boolean lattice  $B_3$ , see Figure 2, which is the ideal lattice of the poset consisting of an anti-chain with three elements.

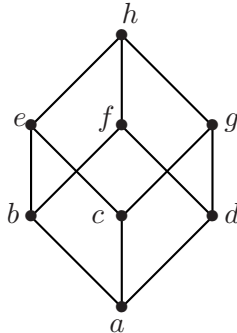


FIGURE 2.

The two Hibi relations  $z_e z_d - z_a z_h$ ,  $z_g z_b - z_a z_h$  can be replaced by the relations  $z_e z_d - z_g z_b$ ,  $z_g z_b - z_a z_h$  where the first of them is not a Hibi relation. Hence in this example, the Hibi relations are not indispensable.

We need some preparations to prove the main theorem of this section.

**Lemma 1.2.** *Let  $L$  be distributive lattice and  $f = z_a z_b - z_c z_d$  be a non-zero element in  $I_L$ . Then  $a \wedge b = c \wedge d$  and  $a \vee b = c \vee d$ . In particular, if  $c$  and  $d$  are comparable, then  $f$  is a Hibi relation.*

*Proof.* For a monomial  $u \in S = K[x_1, \dots, x_n, y_1, \dots, y_n]$  we set

$$\text{supp}_x(u) = \{x_i: x_i \text{ divides } u\} \quad \text{and} \quad \text{supp}_y(u) = \{y_i: y_i \text{ divides } u\}.$$

Since  $f \in \text{Ker}(\varphi)$ , we have

$$\text{supp}_x(u_a u_b) = \text{supp}_x(u_c u_d) \quad \text{and} \quad \text{supp}_y(u_a u_b) = \text{supp}_y(u_c u_d),$$

where for  $e \in L$ ,  $u_e$  denotes the monomial defined as in (1).

This implies that  $a \wedge b = c \wedge d$  and  $a \vee b = c \vee d$ .  $\square$

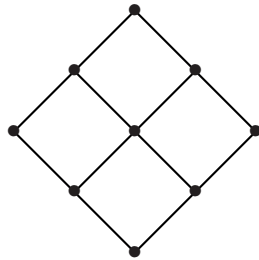
In order to formulate the main result of this section we have to introduce some notation and concepts. Let  $L$  be a lattice and  $[a, b]$  be an interval of  $L$  and  $c, d \in [a, b]$ . Then  $d$  is called a *complement* of  $c$  with respect to  $[a, b]$  if  $d \vee c = b$  and  $d \wedge c = a$ . The set  $\{c, d\}$  is called a *complementary set* of  $[a, b]$ , if  $\{c, d\} \neq \{a, b\}$ . An interval is *complemented* if it admits a complementary set.

**Lemma 1.3.** *Let  $L$  be a distributive lattice,  $[a, b]$  an interval of  $L$  and  $c \in [a, b]$ . Suppose  $c$  has a complement with respect to  $[a, b]$ , then this complement is uniquely determined.*

*Proof.* The proof follows from the fact a distributive lattice does not contain a sublattice as shown in Figure 1.  $\square$

We call a lattice  $L$  *uniquely relatively complemented* or a *URC-lattice* if for every interval  $[a, b]$  of  $L$  either  $[a, b]$  is a chain or there exists a unique complementary set  $\{c, d\}$  of  $[a, b]$ . The lattice  $L$  is said to be a *conditionally URC-lattice*, if for each interval  $[a, b]$  of  $L$ , a complementary set of  $[a, b]$  is unique provided it exists.

The following figures show an example of a URC-lattice and a conditionally URC-lattice.



URC Lattice



Conditionally URC lattice

FIGURE 3.

**Theorem 1.4.** *A URC lattice is distributive.*

*Proof.* The proof follows from the fact that a URC lattice does not contain any sublattice shown in Figure 1.  $\square$

In the case that  $L$  is a distributive lattice, the conditionally URC property can be characterized as follows.

**Lemma 1.5.** *Let  $L$  be a distributive lattice  $L$ . Then the following conditions are equivalent:*

- (a) *For all  $y \in L$ ,  $y$  has at most two lower neighbors.*
- (b) *For all  $x \in L$ ,  $x$  has at most two upper neighbors.*
- (c)  *$L$  is conditionally URC.*

*Proof.* (a) $\Rightarrow$ (b): Suppose  $x \in L$  has three distinct upper neighbors, say,  $l, m, n$ . Since  $L$  is distributive, it follows that  $l \vee n \vee m$  has at least three distinct lower neighbors, namely,  $l \vee m$ ,  $l \vee n$  and  $m \vee n$ . This leads to contradiction to our assumption.

(b) $\Rightarrow$ (a) is proved similarly.

(b) $\Rightarrow$ (c): Suppose  $L$  is not conditionally URC. Then there exists an interval  $[a, b]$  of  $L$  such that it has two distinct complementary sets  $\{c_1, c_2\}$  and  $\{d_1, d_2\}$ . It follows from Lemma 1.3, that  $\{c_1, c_2\} \cap \{d_1, d_2\} = \emptyset$ .

Assume that one of the  $c_i$  is comparable with one of the  $d_j$ , say,  $c_1 < d_2$ . Then  $c_1 \wedge d_1 = a$ , because  $a \leq c_1 \wedge d_1 \leq d_2 \wedge d_1 = a$ . Then  $c_1 \vee d_1 < b$ . Let  $b_1$  and  $b_2$  be the two lower neighbors of  $b$ , and  $a_1$  and  $a_2$  be the two upper neighbors of  $a$ . We may assume that  $d_1 < b_1$  and  $d_2 < b_2$ . We have  $c_1 \vee d_1 \leq b_1 < b$  which implies  $c_1 < b_1$ . Since we assume that  $c_1 < d_2$ , we also get  $c_1 < b_2$ . On the other hand,  $c_2 < d_1$  or  $c_2 < d_2$ , which gives  $c_1 \vee c_2 < b$ , a contradiction.

So,  $c_1, c_2, d_1, d_2$  are pairwise incomparable. We may assume that  $c_1, d_1 < b_1$  and  $c_2, d_2 < b_2$ . Clearly,  $c_1 \vee d_2 = b$ . It follows from Lemma 1.3 that  $c_1 \wedge d_2 > a$ . We can assume that  $c_1 \wedge d_2 \geq a_1 > a$  which gives  $c_1, d_2 \geq a_1$  and  $c_2, d_1 \geq a_2$ . This implies that  $c_1 \wedge d_1 = a$ , since  $c_1 \not\geq a_2$  and  $d_1 \not\geq a_1$ . Distributivity of  $L$  gives  $d_1 = (c_1 \vee d_2) \wedge d_1 = (c_1 \wedge d_1) \vee (d_2 \wedge d_1) = a$ , a contradiction.

(c) $\Rightarrow$ (a): Suppose there exists  $x \in L$  such that  $x$  has at least three lower neighbors, say,  $a, b, c$ . Since  $L$  is distributive it follows that

$$a \wedge b \neq b \wedge c \neq c \wedge a.$$

The sets  $\{a \wedge b, c\}$ ,  $\{b \wedge c, a\}$  are distinct complementary sets of interval  $[a \wedge b \wedge c, x]$ , a contradiction.  $\square$

For an integer  $k \geq 0$ , we set  $[k]_0 = \{0, 1, \dots, k\}$ . Now we can state the main result of this section.

**Theorem 1.6.** *Let  $P$  be a finite poset and  $L$  its ideal lattice. The following conditions are equivalent:*

- (a) *For  $I_L$  the Hibi relation are indispensable.*
- (b)  *$L$  is conditionally URC.*
- (c) *In the poset  $P$ , all poset ideals are generated by at most 2 elements.*

- (d) The poset  $P$  can be covered by two disjoint chains, i.e., we have chains  $C$  and  $D$  in  $P$  such that  $V(P) = V(C) \cup V(D)$  and  $V(C) \cap V(D) = \emptyset$ .
- (e)  $L$  can be embedded as a full sublattice in  $[m]_0 \times [n]_0$ , where  $m = |C|$  and  $n = |D|$ .

*Proof.* (a) $\Rightarrow$ (b): Suppose that  $L$  is not conditionally URC. Then there exist an interval  $[a, b]$  of  $L$  such that it has two distinct complementary sets  $\{x, y\}$  and  $\{r, s\}$ . For these two sets, we have two Hibi relations  $h_1 = z_x z_y - z_a z_b$  and  $h_2 = z_r z_s - z_a z_b$  in  $I_L$  which implies that  $h_3 = z_x z_y - z_r z_s \in I_L$ . The relation  $h_3$  is not a Hibi relation and  $h_1 = h_2 + h_3$ . It shows that  $h_1$  is dispensable.

(b) $\Rightarrow$ (a): Let  $L$  be a conditionally URC lattice and  $H$  be the set of all Hibi relations in  $I_L$ . Take  $f \in H$  where  $f = z_c z_d - z_a z_b$  and  $\{c, d\}$  is a complementary set of  $[a, b]$ . Suppose  $f$  is dispensable. Then it can be written as a  $K$ -linear combination of some other degree 2 binomials  $g_1, \dots, g_n$  in  $I_L$  with  $g_i \neq f$  for all  $i$ . It follows that  $z_a z_b \in \text{supp } g_i$  for some  $i \in [n]$ , say,  $g_i = z_r z_s - z_a z_b$ . From Lemma 1.2, we know that  $g_i$  must be a Hibi relation, i.e.,  $r \wedge s = a$  and  $r \vee s = b$ . Since  $L$  is conditionally URC, we must have  $\{c, d\} = \{r, s\}$ . It gives  $f = g_i$ , a contradiction.

(b) $\Rightarrow$ (c): Suppose there exists a poset ideal  $(p, q, r)$  of  $P$  which is minimally generated by three elements. Clearly,  $p, q$  and  $r$  are incomparable in  $P$ . Let  $b = (p, q, r)$ . Then  $b$  has three lower neighbors in  $L$ , namely  $b/\{r\}$ ,  $b/\{p\}$  and  $b/\{q\}$ , which contradicts Lemma 1.5.

(c) $\Rightarrow$ (d): We choose a chain of ideals  $\emptyset = a_0 \subset a_1 \subset a_2 \subset \dots \subset a_s = P$  with  $\sharp(a_i \setminus a_{i-1}) = 1$ , for all  $i$ . Each  $a_i$  may be viewed as subposet of  $P$  which also satisfies condition (c). Thus by induction on the cardinality of the poset we may assume that  $a_{s-1}$  can be covered by two disjoint chains, say  $C_0$  and  $D_0$  with maximal elements  $q$  and  $r$  respectively. Take  $p \in P$  such that  $a_s = a_{s-1} \cup \{p\}$ .

Suppose that  $p$  is comparable with either  $q$  or  $r$ , say comparable with  $q$ . Then we let  $C = C_0 \cup \{p\}$  and  $D = D_0$ . Otherwise we may assume that there exist a lower neighbor of  $p$  in  $D_0$  different from  $r$ . Let  $D_0 = \{d_1, d_2, \dots, d_k\}$  with  $d_1 < d_2 < \dots < d_k$ . Suppose that the lower neighbor of  $p$  in  $D_0$  is  $d_i$  with  $i < k$ . It follows that  $d_{i+1}$  is comparable with  $q$ , because otherwise  $(p, q, d_{i+1})$  is a 3-generated ideal, contradicting our assumption (c). In both cases, namely  $q < d_{i+1}$  and  $q > d_{i+1}$ , we define  $C = C_0 \cup \{d_{i+1}, \dots, d_k\}$  and  $D = \{d_1, \dots, d_i, p\}$ . Note that, if  $q > d_{i+1}$ , then  $C_0 \cup \{d_{i+1}, \dots, d_k\}$  is a chain. Otherwise, for any  $c_i$  incomparable with some  $d_{i+l}$  and  $i + l < k$ , we have  $c_i$  incomparable with  $p$ , because  $c_i < p$  gives  $c_i \leq d_i < d_{i+l}$ . Then the ideal  $(c_i, d_{i+l}, p)$  is 3-generated ideal, a contradiction.

(d) $\Rightarrow$ (e): Let  $C$  and  $D$  be given by  $c_1 < \dots < c_n$  and  $d_1 < \dots < d_m$  respectively. We define the embedding  $\varphi: L \rightarrow \mathbb{N}^2$  by

$$\varphi(a) = \begin{cases} (i, j), & \text{if } a \cap C = (c_i) \text{ and } a \cap D = (d_j), \\ (0, j), & \text{if } a \cap C = \emptyset \text{ and } a \cap D = (d_j), \\ (i, 0), & \text{if } a \cap C = (c_i) \text{ and } a \cap D = \emptyset, \\ (0, 0), & \text{if } a \cap C = \emptyset \text{ and } a \cap D = \emptyset. \end{cases}$$

Observe first that  $\varphi$  is injective. Indeed, if  $\varphi(a) = \varphi(b)$ , then  $a \cap C = b \cap C$  and  $a \cap D = b \cap D$ . Since  $P = C \cup D$ , we then have

$$a = a \cap P = a \cap (C \cup D) = (a \cap C) \cup (a \cap D) = (b \cap C) \cup (b \cap D) = b \cap (C \cup D) = b \cap P = b.$$

Next we show that  $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ . Let  $\varphi(a) = (i, j)$  and  $\varphi(b) = (k, l)$ . Then,

$$((a \wedge b) \cap C, (a \wedge b) \cap D) = ((a \cap C) \cap (b \cap C), (a \cap D) \cap (b \cap D)) = (c_{\min\{i,k\}}, d_{\min\{j,l\}}).$$

Therefore,  $\varphi(a \wedge b) = (\min\{i, k\}, \min\{j, l\}) = \varphi(a) \wedge \varphi(b)$ . For the join the argument is similar.

Now it remains to be shown that the embedding yields a full sublattice of  $[m]_0 \times [n]_0$ , where  $n = |C|$  and  $m = |D|$ . In other words we have to show that  $\varphi(L)$  contains a chain of length  $n + m$ . For this consider the chain of ideals in  $P$  which we introduced in the proof (c)  $\Rightarrow$  (d). By construction, this chain has length  $|P| = n + m$ . Therefore  $\varphi(a_0) < \varphi(a_1) < \dots < \varphi(a_{n+m})$  is the desired chain in  $\varphi(L)$ .

(e) $\Rightarrow$ (b): Let  $(i, j) \in L$ . Since  $L$  is full sublattice of  $[m]_0 \times [n]_0$ , it follows that each upper neighbor of  $(i, j)$  is of the form  $(i + 1, j)$  or  $(i, j + 1)$ . So the assertion follows from Lemma 1.5.  $\square$

An interesting special case of the previous theorem is described in the next result.

**Proposition 1.7.** *Let  $P$  be a finite poset and  $L$  be its ideal lattice. Then following conditions are equivalent.*

- (a)  $L$  is a URC lattice.
- (b) Either  $P$  is a chain or it consists of two disjoint chains  $C$  and  $D$  such that all elements of  $C$  are incomparable with all elements of  $D$ .
- (c) There exist non-negative integers  $m$  and  $n$  such that  $L \cong [m]_0 \times [n]_0$ .

*Proof.* (a) $\Rightarrow$ (b): From Theorem 1.6, we know that there exist two disjoint chains  $C$  and  $D$  which cover  $P$ . Assume that  $P$  does not satisfy (b). Then  $P$  contains two incomparable elements, say  $p_1 \in C$  and  $p_2 \in D$ . Moreover, there exist  $c \in C$  and  $d \in D$  such that they are comparable. We may assume that  $c_i > d_j$ .

Suppose that  $P$  has only one minimal element, say  $q$ . The interval  $[\emptyset, (p_1, p_2)]$  of  $L$  is not a chain because it contains two incomparable elements  $(p_1)$  and  $(p_2)$ . Moreover, this interval does not have a complementary set because the only upper neighbor of  $\emptyset$  in  $L$  is  $(p)$ , a contradiction.

Now suppose that  $P$  has two minimal elements, say  $q_1 \in C$  and  $q_2 \in D$ . It follows that  $c > q_1, q_2$ . Let  $c'$  be the minimal element in  $C$  with this property. Then  $c'$  has two incomparable lower neighbors  $r_1$  and  $r_2$  in  $P$ . Therefore it follows that the interval  $[(r_1) \cap (r_2), (c')]$  of  $L$  is not a chain and does not have a complementary set, because  $(r_1, r_2)$  is the only lower neighbor of  $(c')$  in  $[(r_1) \cap (r_2), (c')]$ , again a contradiction.

(b)  $\Rightarrow$  (c): If  $P$  is a chain then  $L \cong [m]_0 \times [0]_0$ . Otherwise,  $P$  is the disjoint union of two chains  $C : c_1 < c_2 < \dots < c_m$  and  $D : d_1 < d_2 < \dots < d_n$ , where none of the  $c_i$  is comparable with any of the  $d_j$ . As in the proof of (d)  $\Rightarrow$  (c) of Theorem 1.6, we have the embedding  $\varphi : L \rightarrow [m]_0 \times [n]_0$ . To show that  $\varphi$  is an isomorphism it is enough to show that  $|L| = (m + 1)(n + 1)$ . To see this we observe that if  $\alpha \in L$  then  $\alpha = \emptyset$  or  $\alpha = (c_i)$  or  $\alpha = (d_j)$  or  $\alpha = (c_i, d_j)$ . It is obvious that ideals  $\emptyset, (c_i), (d_j)$  are pairwise distinct, and that these ideals are also different from the 2-generated ideal  $(c_i, d_j)$ . Suppose now that  $(c_i, d_j) = (c_k, d_l)$ . Since the elements of  $C$  are all incomparable with elements of  $D$ , it follows that  $c_i \leq c_k$  and  $d_j \leq d_l$ . Similarly one has  $c_k \leq c_i$  and  $d_l \leq d_j$ . Altogether we conclude that  $|L| = (m + 1)(n + 1)$ .



(c) $\Rightarrow$ (a): Let  $L \cong [m]_0 \times [n]_0$  for some non-negative integers  $m$  and  $n$ . To show that  $L$  is indeed a URC lattice, it is enough to show that every interval in  $L$  which is not a chain has a complementary set. Let  $[(i, j), (k, l)]$  be an interval in  $L$  with  $i < k$  and  $j < l$ . There exist two incomparable elements  $a, b \in L$ , namely  $a = (k, j)$  and  $b = (i, l)$  with  $a \wedge b = (i, j)$  and  $a \vee b = (k, l)$ .  $\square$

## 2. GRÖBNER BASES OF HIBI RINGS WITH RESPECT TO RANK LEXICOGRAPHIC ORDERS

In this section we want to classify all distributive lattices with the property that with respect to the rank lexicographic order the Hibi ideal of the lattice has a reduced Gröbner basis consisting of Hibi relations.

In order to formulate our main result we introduce some terminology. Let  $L$  be a full sublattice of  $[m]_0 \times [n]_0$ . Let  $(i, j)$  be an element in  $L$  such that  $(i-1, j), (i+1, j), (i, j+1), (i, j-1)$  also belong to  $L$ . We call it an *upper corner* if  $(i-1, j+1) \notin L$  and  $(i+1, j-1) \in L$ , a *lower corner* if  $(i-1, j+1) \in L$  and  $(i+1, j-1) \notin L$  and *critical corner* if  $(i-1, j+1) \notin L$  and  $(i+1, j-1) \notin L$ . A lattice  $L$  is called a *chain ladder*, (see [2]), if all upper corners and lower corners appear in a chain and that, for any two corners  $(i, j) \neq (i', j')$  of  $D$ , one has  $i \neq i'$  and  $j \neq j'$ .

**Theorem 2.1.** *Let  $L$  be a distributive lattice. The following conditions are equivalent:*

- (a) *The reduced Gröbner basis of  $I_L$  with respect to a rank lexicographic order consists of all Hibi relations in  $I_L$ .*
- (b) *The Hibi relations are indispensable, and  $I_L$  has a reduced quadratic Gröbner basis with respect to a rank lexicographic order.*
- (c)  *$L$  is conditionally URC, and for all  $a < b < c$  in  $L$  such that  $[a, b]$  and  $[b, c]$  have complementary sets, it follows that either  $[g, c]$  or  $[h, c]$  is complemented, where  $\{g, h\}$  is the complementary set of  $[a, b]$ .*
- (d)  *$L$  is isomorphic to a chain ladder without critical corners.*

*Proof.* (a) $\Rightarrow$ (b): We have a quadratic Gröbner basis since Hibi relations are quadratic. Suppose that  $f$  is a quadratic binomial relation with  $\text{in}(f) = z_a z_b$ . It follows from (a) that  $a$  and  $b$  are comparable. Therefore Lemma 1.2 implies that  $f = z_a z_b - z_c z_d$ , where  $[c, d]$  is complementary pair of  $[a, b]$ , as desired.

(b) $\Rightarrow$ (a): Let  $f$  be a binomial in reduced Gröbner basis of  $I_L$ . By our assumption  $f$  is a quadratic binomial. Since Hibi relations are indispensable  $f$  must be a Hibi relation.

(b) $\Rightarrow$ (c): From Theorem 1.6, we know that  $L$  is conditionally URC and it can be identified with a full sublattice in  $[m]_0 \times [n]_0$ . Let  $a, b$  and  $c$  be the elements in  $L$  such that  $[a, b]$  and  $[b, c]$  are complemented with complementary pairs  $\{g, h\}$ , and  $\{d, e\}$ , respectively.

Consider the S-polynomial  $z_c z_g z_h - z_a z_e z_d$  of the Hibi relations  $z_a z_b - z_g z_h$  and  $z_b z_c - z_d z_e$ . The monomial  $z_c z_g z_h$  is the leading term of the S-polynomial. Since by our assumption the Gröbner basis of  $I_L$  consists of Hibi relations, it follows that there exists a Hibi relation with initial term  $z_c z_g$  or  $z_c z_h$ . This implies that the interval  $[g, c]$  or  $[h, c]$  is complemented.

(c) $\Rightarrow$ (d): Since  $L$  is a conditionally URC, we may identify it with a full sublattice in  $[m]_0 \times [n]_0$ . Suppose  $L$  has a critical corner  $b = (i, j)$ . By definition of critical corner  $(i-1, j), (i+1, j), (i, j+1), (i, j-1) \in L$ . Therefore, since  $L$  is a lattice,  $a = (i-1, j-1)$  and



$c = (i+1, j+1)$  belong to  $L$ . Let  $[a, b] = [(i-1, j-1), (i, j)]$  and  $[b, c] = [(i, j), (i+1, j+1)]$ , and  $d = (i, j+1)$  and  $e = (i+1, j)$ . Since  $(i-1, j+1) \notin L$  and  $(i+1, j-1) \notin L$ , it follows that  $[a, d]$  and  $[a, e]$  are not complemented, a contradiction.

It remains to show that  $L$  is a chain ladder. First, suppose that  $L$  has two incomparable corners,  $x = (i, j)$  and  $y = (k, l)$ . Then we may assume  $i < k$ ,  $j > l$ . Since  $L$  is a lattice it contains also the elements  $w = x \wedge y = (i, l)$  and  $z = x \vee y = (k, j)$  and since  $L$  is a full sublattice of  $[m]_0 \times [n]_0$ , it contain all elements  $\{(r, s) : i \leq r \leq k, l \leq s \leq j\}$ . This implies  $x$  is an upper corner and  $y$  is a lower corner. By definition of corners, it follows that  $d = (i-1, j)$ ,  $e = (i, j+1)$ ,  $g = (k+1, l)$  and  $f = (k-1, l)$  belong to  $L$ . Hence  $a = d \wedge f = (i-1, l-1)$  and  $c = e \vee g = (k+i, j+i)$  belong to  $L$ . Now we have  $a < b < c$ . The interval  $[a, b]$ ,  $[b, c]$  are complemented. Therefore, either the interval  $[d, c]$  and  $[f, c]$  must be complemented by our assumption (c), contradicting the fact that  $x$  and  $y$  are upper and lower corners respectively.

Now suppose  $L$  has two corners  $a = (i, j)$  and  $b = (k, l)$  such that either  $i = k$  or  $j = l$ . Let  $j = l$ . We can assume that  $i < k$ . It gives  $a < b$ . By the definition of corners, the elements  $(i-1, j)$ ,  $(i, j-1)$ ,  $(i+1, j)$ ,  $(i, j+1)$  and  $(k+1, j)$ ,  $(k, j+1)$ ,  $(k-1, j)$ ,  $(k, j-1)$  belong to  $L$ . Since  $L$  is a full sublattice of  $[m]_0 \times [n]_0$ , it follows that  $[(i, j-1), (k, j-1)] \subset L$ . In particular  $(i+1, j-1) \in L$ . This shows  $a$  is an upper corner. Similarly one shows that  $b$  is a lower corner. Since  $L$  is a lattice  $c = (i-1, j-1)$  and  $d = (k+1, j+1)$  also belong to  $L$ . We have  $c < b < d$  and also the intervals  $[c, b]$  and  $[b, d]$  are complemented. From (c), we know that either  $[(k, j-1), d]$  or  $[(i-1, j+1), d]$  must be complemented, in other words, either  $(i-1, j+1)$  or  $(k+1, j-1)$  must belong to  $L$ . This contradicts our supposition. A similar argument holds if we assume  $i = k$ .

(d) $\Rightarrow$ (b): It is shown [1, Theorem 2.5] that  $I_L$  has a quadratic Gröbner basis under the additional assumption that  $L$  is simple. In the same way it is shown that  $L$  has quadratic Gröbner basis even if it is not simple, provided it satisfies (d). Since  $L$  is a conditionally URC, it follows from Lemma 1.6, that Hibi relations are indispensable.  $\square$

### 3. REES RINGS OF HIBI IDEALS

Let  $L$  be the ideal lattice of the poset  $P = \{p_1, \dots, p_n\}$ , and  $S = K[\{x_{p_i}, y_{p_i}\}_{p_i \in P}]$  be the polynomial ring in  $2n$  variables over a field  $K$  with  $\deg x_{p_i} = \deg y_{p_i} = 1$ . Recall that to each element  $a \in L$ , we associate a squarefree monomial  $u_a = \prod_{p_i \in a} x_{p_i} \prod_{p_i \notin a} y_{p_i}$  and the Hibi ideal  $H_L$  is defined to be the ideal of  $S$  generated by such monomials, i.e.  $H_L = (u_a | a \in L)$ , see [4].

Let  $\mathcal{R}(H_L)$  denote the Rees algebra of  $H_L$  and  $J_L$  be the defining ideal of  $\mathcal{R}(H_L)$ . In other words,  $\mathcal{R}(H_L)$  is the affine semigroup ring given by

$$\mathcal{R}(H_L) = S[\{u_a t\}_{a \in L}] = K[\{x_{p_i}, y_{p_i}\}_{p_i \in P}, \{u_a t\}_{a \in L}] \subset K[\{x_{p_i}, y_{p_i}\}_{p_i \in P}, t],$$

and  $J_L$  is the kernel of the surjective ring homomorphism  $\varphi : R \rightarrow \mathcal{R}(H_L)$  where

$$R = S[\{z_a\}_{a \in L}] = K[\{x_{p_i}, y_{p_i}\}_{p_i \in P}, \{z_a\}_{a \in L}]$$

is a polynomial ring over  $K$  and  $\varphi$  is defined by setting

$$(2) \quad \varphi(x_{p_i}) = x_{p_i}, \varphi(y_{p_i}) = y_{p_i}, \varphi(z_a) = u_a t.$$

In this section we are interested in the Gröbner basis of  $J_L$  with respect to a suitable lexicographical orders. We define a term order on  $R = K[\{x_{p_i}, y_{p_i}\}_{p_i \in P}, \{z_a\}_{a \in L}]$  and for the sake of convenience we write  $x_i, y_i$  instead of  $x_{p_i}, y_{p_i}$ . The term order on  $R$ , denoted by  $<_{lex}^1$ , is defined to be the product order of the lexicographic order on  $S$  induced by  $x_1 > \cdots > x_n > y_1 > \cdots > y_n$  and a rank lexicographical order on  $T$ . In particular  $x_i >_{lex}^1 y_j >_{lex}^1 z_a$  for all  $i, j$  and  $a$ .

Let  $a_1$  and  $a_2$  be two poset ideals of  $P$  such that  $a_2 = a_1 \cup \{p_i\}$ . To each such pair of poset ideals, we associate a binomial  $x_i z_{a_1} - y_i z_{a_2}$ , and call it a *special linear relation* in  $R$ .

Now we state the main theorem of this section.

**Theorem 3.1.** *Let  $L$  be a distributive lattice. Then following conditions are equivalent.*

- (a)  $L$  is a URC lattice.
- (b) The reduced Gröbner basis of  $J_L$  with respect to  $<_{lex}^1$  consists of Hibi relations and special linear relations.

*Proof.* (a)  $\Rightarrow$  (b): From [4, Theorem 1.1] and its proof, we know that  $J_L$  is minimally generated by Hibi relations and special linear relations. Let  $M$  be the set of these relations. To show that  $M$  is a reduced Gröbner basis of  $J_L$  with respect to  $<_{lex}^1$ , we must show that all S-pairs  $S(f_i, f_j)$ ,  $1 \leq i, j \leq n$  reduce to 0. Take  $f_i, f_j \in M$  and consider the non-trivial case when  $\gcd(\text{in}_<(f_i), \text{in}_<(f_j)) \neq 1$ . For any binomial, we always write the leading term as the first term.

If  $f_i$  and  $f_j$  are both Hibi relation then  $S(f_i, f_j)$  reduces to 0 because of Theorem 2.1. Next we consider the case that  $f_i$  is a Hibi relation and  $f_j$  is a special linear relation. Say,

$$f_i = z_d z_a - z_b z_c \quad \text{with} \quad d > a, \quad \text{and} \quad f_j = x_p z_a - y_p z_e \quad \text{or} \quad f_j = x_p z_d - y_p z_e.$$

Let us first assume that  $f_j = x_p z_a - y_p z_e$ . Then it follows from the relation  $f_j$  that  $a$  is a lower neighbor of  $e$ . From Proposition 1.7, we know that  $L \cong [m]_0 \times [n]_0$ . Let  $b = (i, j)$  and  $c = (k, l)$  with  $i < k$  and  $j > l$ . Then  $a = (i, l)$  and  $d = (k, j)$ . Since  $a$  is a lower neighbor of  $e$ , we have  $e = (i, l + 1)$  or  $e = (i + 1, l)$ . Assume  $e = (i, l + 1)$ . Take  $f = (k, l + 1)$ . Then  $c$  is a lower neighbor of  $f$ , see Figure 4.

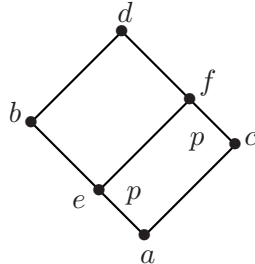


FIGURE 4.

If  $b = e$ , then we also have  $d = f$  and we obtain

$$S(f_i, f_j) = x_p z_b z_c - y_p z_d z_e = z_b (x_p z_c - y_p z_d).$$

Therefore  $S(f_i, f_j)$  reduces to 0.

Now, if  $b > e$ , then we first observe  $\{b, f\}$  is the complementary set in  $[e, d]$ . Therefore, in this case

$$S(f_i, f_j) = x_p z_b z_c - y_p z_d z_e = z_b(x_p z_c - y_p z_f) - y_p(z_d z_e - z_b z_f).$$

It shows that  $S(f_i, f_j)$  again reduces to zero.

Next assume that  $f_j = x_p z_d - y_p z_e$ . It follows from the relation  $f_j$  that  $d$  is lower neighbor of  $e$ . Let  $b = (i, j)$  and  $c = (k, l)$  with  $i < k$  and  $j > l$ . Then  $a = (i, l)$  and  $d = (k, j)$  and either  $e = (k, j + 1)$  or  $e = (k + 1, j)$ . We can assume that  $e = (k + 1, j)$ . Since the interval  $[a, e]$  has the complementary set  $\{b, g\}$ , the interval  $[c, e]$  has the complementary set  $\{d, g\}$  where  $g = (k + 1, l)$ , see Figure 5.

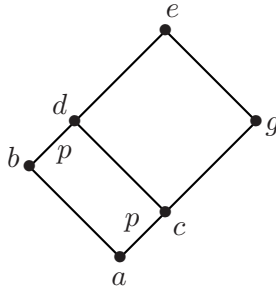


FIGURE 5.

Therefore, we have

$$S(f_i, f_j) = x_p z_b z_c - y_p z_e z_a = z_b(x_p z_c - y_p z_g) - y_p(z_e z_a - z_g z_b)$$

Again,  $S(f_i, f_j)$  reduces to 0.

Now, we consider the case when both  $f_i$  and  $f_j$  are special linear relations. Say,

$$f_i = x_p z_a - y_p z_b \quad \text{and} \quad f_j = x_q z_a - y_q z_c \quad \text{or} \quad f_j = x_p z_d - y_p z_e$$

First assume that  $f_j = x_q z_a - y_q z_c$ . Let  $d = b \vee c$ , see Figure 6.

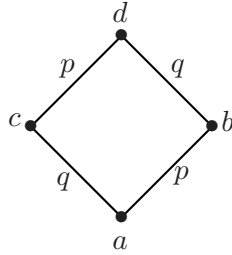


FIGURE 6.

Then  $S(f_i, f_j) = x_p y_q z_c - x_q y_p z_b = y_q(x_p z_c - y_p z_d) - y_p(x_q z_b - y_q z_d)$ . Therefore,  $S(f_i, f_j)$  reduces to 0.

Now, take  $f_j = x_p z_d - y_p z_e$ . We can assume that  $b > e$ . Take  $a = (i, j)$ ,  $b = (i + 1, j)$ ,  $d = (i, l)$  and  $e = (i + 1, l)$  where  $j > l$ , see the Figure 7.

Then  $\{a, e\}$  is the complementary set in  $[d, b]$  and we have

$$S(f_i, f_j) = y_p z_a z_e - y_p z_b z_d = -y_p(z_b z_d - z_a z_e)$$

Hence  $S(f_i, f_j)$  reduces to 0. This complete the proof.

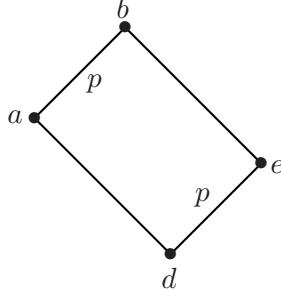


FIGURE 7.

(b) $\Rightarrow$ (a): Since  $<_{lex}^1$  is an elimination order for the variables  $x_i$  and  $y_j$ , it follows that the Gröbner basis of  $J_L \cap T$  with respect to the rank lexicographic order consists of elements of the Gröbner basis of  $J_L$  with respect to  $<_{lex}^1$  which belong to  $T$ . By assumption (b) these relations are exactly the Hibi relations in  $J_L$ . Thus, the Gröbner basis with respect to the rank lexicographical order of the Hibi relation ideal of the Hibi ring  $\mathcal{R}_K(L)$  (which is  $J_L \cap T$ ), consists of Hibi relations. Therefore, from Theorem 2.1, we know that  $L$  is a chain ladder without critical corners. Let  $m$  and  $n$  be the non-negative integers such that  $L$  has an embedding in  $[m]_0 \times [n]_0$  and  $(m, n)$  is the maximal element in  $L$ . Then it is enough to show that  $L$  has no upper or lower corners because then  $L \cong [m]_0 \times [n]_0$ .

Suppose  $L$  has upper or lower corners. Let  $C$  be the maximal chain of upper and lower corners in  $L$  with maximal element  $a$ . Let  $a = (i, j)$  and  $b = (m, n)$ . Then  $[a, b]$  is complemented in  $L$ . Take  $\{c, d\}$  be the complementary set of  $[a, b]$ . We can assume that  $a$  is an upper corner in  $L$ , i.e.,  $(i - 1, j + 1) \notin L$ . Then, the elements  $e = (i - 1, j)$ ,  $g = (i, j - 1)$ ,  $f = (i - 1, j - 1)$ , and  $c = (i, j + 1)$  belong to  $L$ . Consider the  $S$ -polynomial of the binomials  $f_i = z_a z_f - z_e z_g$  and  $f_j = z_a x_p - z_c y_p$  in  $J_L$ , where  $c = a \cup \{p\}$ . Then  $S(f_i, f_j) = x_p z_e z_g - z_f z_c y_p$  reduces to 0 if and only if  $x_p z_e - y_p z_h \in J_L$ , where  $h = (i - 1, j + 1)$ . This implies  $(i - 1, j + 1) \in L$ , a contradiction to our assumption.  $\square$

In the following we extend the previous result to meet-distributive meet-semilattices. Recall that a poset  $L$  is called a *meet-semilattice* if every pair of elements of  $L$  has a meet in  $L$ . A finite meet-semilattice  $L$  is called *meet-distributive* if each interval  $[x, y]$  of  $L$  such that  $x$  is the meet of the lower neighbors of  $y$  in this interval is Boolean. Let  $P$  be the set of join irreducible elements in  $L$ . For any  $l \in L$ , we call the cardinality of  $\{p \in P \mid p \leq l\}$  the *degree* of  $l$ , and the maximum of the lengths of chains descending from  $l$  the *rank* of  $l$ .  $L$  is called *graded* if all maximal chains have the same length. In [3], the following characterization of meet-distributive meet-semilattices is given.

**Lemma 3.2.** *For a finite lattice  $L$  the following conditions are equivalent:*

- (a)  $L$  is meet-distributive.
- (b)  $L$  is graded and  $\deg l = \text{rank } l$ , for all  $l \in L$ .
- (c) Each element in  $L$  is a unique minimal join of join-irreducible elements.

The above lemma shows that a distributive lattice is also a meet-distributive meet-semilattice.

Let  $L$  be a meet-distributive meet-semilattice and  $P$  be the poset consisting of all the join-irreducible elements in  $L$ . We denote by  $\hat{L}$  the ideal lattice of  $P$  and call it *associated distributive lattice* of  $L$ . We have a canonical embedding of  $L$  in  $\hat{L}$  given by  $l \mapsto \{p \in P | p \leq l\}$  for all  $l \in L$ .

**Proposition 3.3.** *Let  $L$  be a meet-distributive meet-semilattice and  $\hat{L}$  be its associated distributive lattice. Then  $L$  is a poset ideal of  $\hat{L}$ .*

*Proof.* Take  $s \in \hat{L}$  and  $r \in L$  such that  $s \leq r$ . From Lemma 3.2, we have  $\text{rank}_L r = \deg_L r$ . Also, we have  $\deg_L r = \deg_{\hat{L}} r = \text{rank}_{\hat{L}} r$ , which gives  $\text{rank}_L r = \text{rank}_{\hat{L}} r$ . It shows any maximal chain descending from  $r$  in  $\hat{L}$  also survives in  $L$ . Hence, we obtain  $s \in L$ .  $\square$

We denote by  $H_L$  the ideal of  $S$  generated by monomials  $u_a$  with  $a \in L$  as described in (1). Let  $\mathcal{R}(H_L)$  denote the Rees algebra of  $H_L$  and  $J_L$  be the defining ideal of  $\mathcal{R}(H_L)$ . We have  $H_L \subset H_{\hat{L}}$  and  $\mathcal{R}(H_L) \subset \mathcal{R}(H_{\hat{L}})$ .

**Corollary 3.4.** *Let  $L$  be a meet-distributive meet-semilattice. Suppose that the associated distributive lattice  $\hat{L}$  of  $L$  is a URC lattice. Then the following conditions are equivalent:*

- (a)  $L = \hat{L}$ .
- (b) *The reduced Gröbner basis of  $J_L$  with respect to  $<_{lex}^1$  consists of Hibi relations and special linear relations.*

*Proof.* (a)  $\Rightarrow$  (b) follows from Theorem 3.1.

(b)  $\Rightarrow$  (a): Assume  $L \subsetneq \hat{L}$ . Since  $L$  is a poset ideal of  $\hat{L}$  and  $\hat{L} \cong [m]_0 \times [n]_0$ , there exist two incomparable elements  $a, b \in L$  such that they cover  $c = a \wedge b$  and  $d = a \vee b \notin L$ . Let  $a = c \cup \{p\}$  and  $b = c \cup \{q\}$  with  $p, q \in P$ . Then  $f_i = x_p z_c - y_p z_a$  and  $f_j = x_q z_c - y_q z_b$  are special linear relations in  $J_L$ , and

$$S(f_i, f_j) = x_p y_q z_b - x_q y_p z_a$$

with the initial monomial  $x_p y_q z_b$  if  $x_p > x_q$ , as we may assume. Our assumption (b) implies that the initial monomial of some Hibi relation or special linear relation must divide  $x_p y_q z_b$ . It follows that the only special linear relation whose initial term divides  $x_p y_q z_b$  is  $x_p z_b - y_p z_d$ . Since  $d \notin L$ , we arrive at a contradiction.  $\square$

## REFERENCES

- [1] A. Aramova, J. Herzog, T. Hibi, *Finite lattices and lexicographic Gröbner bases*, Europ. J. Combinatorics **21** (2000), 431–439.
- [2] A. Conca, J. Herzog, *Ladder determinantal rings have rational singularities*, Adv. Math. **132** (1997), 120 – 147.
- [3] P. Edelman, *Abstract convexity and meet-distributive lattices*, in Combinatorics and ordered sets (Arcata, Calif., 1985), Contemp. Math. **57**, 127–150, Amer. Math. Soc., Providence, RI, 1986.
- [4] J. Herzog, T. Hibi, *Distributive Lattices, Bipartite Graphs and Alexander Duality*, J. Alg. Comb. **22** (2005), 289–302.
- [5] J. Herzog, T. Hibi, X. Zheng, *The monomial ideal of a finite meet-semilattice*, The Transaction AMS, **358** (2006), 4119–4134.
- [6] J. Herzog, T. Hibi, *Monomial Ideals*, GTM 260, Springer, 2010.
- [7] T. Hibi, *Distributive lattices, affine semigroups rings and algebra with straightening laws*, in Commutative Algebra and Combinatorics (M. Nagata and H. Matsumura, eds.) Adv. Stud. Pure Math. **11**, 93–109, North-Holland, Amsterdam, 1987.

- [8] H. Ohsugi, T. Hibi, *Indispensable binomials of finite graphs*, J. Algebra Appl. **4** (2005), 421-434.
- [9] R.P. Stanley, Enumerative Combinatorics, Volume 1, Cambridge, 1997.

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